MATHEMATICS

ALGEBRAIC INDEPENDENCE OF CERTAIN NUMBERS IN THE P-ADIC DOMAIN

BY

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§ 1. The purpose of this paper is to establish P-adic analogues of the recent results of TIJDEMAN [7] on the algebraic independence of numbers connected with the exponential function.

Let $\mathbf{Z}$ denote the set of rational integers and let $\mathbf{Q}$ denote the field of rational numbers. Let P be a positive prime integer. Let $\mathbf{Q}_P$ be the completion of $\mathbf{Q}$ with respect to the P-adic valuation $||_P$ on $\mathbf{Q}$. Let T_P be the completion of the algebraic closure of $\mathbf{Q}_P$ and denote its valuation by $||_P$ also.

THEOREM 1. *Let $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in T_P$ with $|\alpha_i|_P < P^{-1/P-1}$ be linearly independent over $\mathbf{Q}$. Let $\eta_0, \eta_1, \eta_2 \in T_P$ with $|\eta_i|_P \leqslant 1$ be linearly independent over $\mathbf{Q}$. Then at least two of the numbers*

$$\alpha_k, \exp(\alpha_k \eta_J), \quad k = 0, 1, 2, 3 \quad J = 0, 1, 2$$

are algebraically independent over $\mathbf{Q}$.

THEOREM 2. *Let $\alpha_0, \alpha_1, \alpha_2 \in T_P$ with $|\alpha_i|_P < P^{-1/P-1}$ be linearly independent over $\mathbf{Q}$. Let $\eta_0, \eta_1 \in T_P$ with $|\eta_i|_P \leqslant 1$ be linearly independent over $\mathbf{Q}$. Then at least two of the numbers*

$$\alpha_k, \eta_J, \exp(\alpha_k \eta_J), \quad k = 0, 1, 2 \quad J = 0, 1$$

are algebraically independent over $\mathbf{Q}$.

THEOREM 3. *Let $\alpha_0, \alpha_1, \alpha_2 \in T_P$ with $|\alpha_i|_P < P^{-1/P-1}$ be linearly independent over $\mathbf{Q}$. Let $\eta_0, \eta_1, \eta_2, \eta_3 \in T_P$ with $|\eta_i|_P \leqslant 1$ be linearly independent over $\mathbf{Q}$. Then at least two of the numbers*

$$\alpha_k, \exp(\alpha_k \eta_J), \quad k = 0, 1, 2 \quad J = 0, 1, 2, 3$$

are algebraically independent over $\mathbf{Q}$.

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The theorems (1–3) have several consequences. We mention:

COROLLARY 1. Let $\beta \in T_P$, $|\beta|_P \leqslant 1$, be algebraic of degree $\geqslant 3$. Then at least two of the numbers

$$e^P,\ e^{P\beta},\ e^{P\beta^2},\ e^{P\beta^3}\quad (P>2)$$

are algebraically independent. In particular, if β is algebraic of degree 3, then at least two of the numbers

$$e^P,\ e^{P\beta},\ e^{P\beta^2}\quad (P>2)$$

are algebraically independent. The same type of result holds when $P=2$. (In theorem 2, set $\alpha_k = P\beta^k$, $\eta_J = \beta^J$).

COROLLARY 2. At least two of the numbers

$$e^P,\ e^{Pe^P},\ e^{Pe^{2P}},\ e^{Pe^{3P}}\quad (P>2)$$

are algebraically independent. In particular at least one of the numbers

$$e^{Pe^P},\ e^{Pe^{2P}},\ e^{Pe^{3P}}\quad (P>2)$$

is transcendental. The same type of result holds when $P=2$. (In theorem 2, set $\alpha_k = Pe^{kP}$, $\eta_J = e^{JP}$).

COROLLARY 3. Let $t \in T_P$, $|t|_P \leqslant 1$, be transcendental (or algebraic of degree $\geqslant 3$). Let $\alpha \in T_P$, $\alpha \neq 1$, $|\alpha - 1|_P < P^{-1/P-1}$ be algebraic. Then at least two of the numbers

$$t,\ \log\alpha,\ \alpha^t,\ \alpha^{t^2},\ \alpha^{t^3}$$

$$(\log\alpha,\ \alpha^t,\ \alpha^{t^2},\ \alpha^{t^3})$$

are algebraically independent. (In theorem 2, set $\alpha_k = t^k \log\alpha$ and $\eta_J = t^J$). In particular, when $t = \log\alpha$, then at least two of the numbers

$$t,\ \alpha^t,\ \alpha^{t^2},\ \alpha^{t^3}$$

are algebraically independent. It follows that at least one of the numbers

$$\alpha^t,\ \alpha^{t^2},\ \alpha^{t^3}$$

is transcendental. In this direction, see [3, p. 8, p. 101], [5, p. 67].

COROLLARY 4. Let $t \in T_P$, $|t|_P < P^{-1/P-1}$, be transcendental (or algebraic of degree $\geqslant 4$). Let α be an element of T_P which is not a root of unity and satisfies $|\alpha - 1|_P < 1$, $|\log\alpha|_P \leqslant 1$. Then at least two of the numbers

$$t,\ \alpha^t,\ \alpha^{t^2},\ \alpha^{t^3},\ \alpha^{t^4},\ \alpha^{t^5},\ \alpha^{t^6}$$

$$(\alpha^t,\ \alpha^{t^2},\ \alpha^{t^3},\ \alpha^{t^4},\ \alpha^{t^5},\ \alpha^{t^6})$$

are algebraically independent. (In theorem 1, set $\alpha_k = t^{k+1}$ and $\eta_J = t^J \log\alpha$). In particular, let $\beta \in T_P$ be an algebraic number satisfying $|\beta - 1|_P <$

$<|t|_P P^{-1/P-1}$ and $\beta \neq 1$. Define $\alpha = \exp((\log \beta)/t)$. Then at least two of the numbers

$$t, \beta^t, \beta^{t^2}, \beta^{t^3}, \beta^{t^4}, \beta^{t^5}$$

$$(\beta^t, \beta^{t^2}, \beta^{t^3}, \beta^{t^4}, \beta^{t^5})$$

are algebraically independent.

Remark. When t is algebraic of degree $\geqslant 4$, a result of the type given in Corollary 4 is already obtained by Adams [1, p. 290]. Notice that the result mentioned in Corollary 4 is independent of the degree of t. When t is algebraic of degree 3, Adams' result [1, p. 290] is stronger than that of Corollary 3.

Corollary 5. Let $b_1, b_2, b_3 \in T_P$ be algebraic numbers satisfying i) $|Pb_i|_P < P^{-1/P-1}$, $i=1, 2, 3$ ii) $1, b_1, b_2, b_3$ are linearly independent over $\mathbf{Q}$. Further let $a_1, a_2, a_3 \in T_P$ be algebraic numbers such that $|a_i - 1|_P < P^{-1/P-1}$, $i=1, 2, 3$ and $\log a_1$, $\log a_2$, $\log a_3$ are linearly independent over $\mathbf{Q}$. Then at least two of the numbers

$$a_i^{Pb_J}, \ i=1, 2, 3$$

$$J=1, 2, 3$$

are algebraically independent. (In theorem 1, set $\alpha_k = Pb_k$ with $b_0 = 1$ and $\eta_J = \log a_{J+1}$).

§ 2. We shall require the following lemmas.

Lemma 1. *Let*

$$y_k = a_{k,1}x_1 + \ldots + a_{k,q}x_q \quad (k=1, \ldots, p)$$

be linear forms with integral rational coefficients and q variables, where $0<p<q$ and suppose that the absolute values of all $a_{k,l}$ are not greater than a given positive integer A; then there exists a non-trivial integral rational solution $x_1, x_2, \ldots, x_q$ of $y_1 = 0, \ldots, y_p = 0$ satisfying

$$|x_k| < 1 + (qA)^{p/q-p} \quad (k=1, \ldots, q).$$

This is due to Siegel [6, p. 35].

Lemma 2. *Let $w_1, w_2, \ldots, w_l \in T_P$ with $|w_i|_P < P^{-((1/P-1)+\varepsilon)}$, where $\varepsilon > 0$ is an arbitrary fixed constant. Let $P_1(z), \ldots, P_l(z)$ be non-zero polynomials of degree $\sigma_1 - 1, \ldots, \sigma_l - 1$ respectively. Define*

$$F(z) = \sum_{k=1}^{1} P_k(z)\, e^{w_k z}$$

$$n = \sum_{k=1}^{1} \sigma_k.$$

Assume that $F(z) \not\equiv 0$. *Then the number of zeros of* $F(z)$ *in* $|z|_P \leqslant 1$ *do not exceed*

$$\frac{90}{\varepsilon \log P} + \frac{30}{\varepsilon}\left(\frac{1}{P-1} + \varepsilon\right)(n-1).$$

(See Appendix).

LEMMA 3. *Let* $w \in T_P$ *and let* f_N *be an unbounded and monotone increasing sequence of positive real numbers such that*

$$\lim_{N\to\infty} \frac{f_{N+1}}{f_N} = 1.$$

Then if for $N \geqslant N_0$, *there exists a non-zero polynomial* $F_N(z)$ *with integral coefficients such that*

$$|F_N(w)|_P < P^{-4000 f_N^2} \text{ and max } (\text{deg. } F_N, \log H(F_N))\,^{1)} \leqslant f_N,$$

then w *is algebraic.*

This can be trivially deduced from a result mentioned in ADAMS [1, Lemma 10].

Let F be a field obtained by adjoining to $\mathbf{Q}$ finitely many points. Let the transcendence degree of F over $\mathbf{Q}$ be one. Then there exists a transcendental number w such that F is a finite extension of $\mathbf{Q}(w)$. Further there exists $w_1 \in F$ which satisfies a monic irreducible polynomial of degree ν (say) over $\mathbf{Z}[w]$ such that $F = \mathbf{Q}(w, w_1)$. By an integer in F, we mean an element of F of the type $\sum_{i=0}^{n} \sum_{J=0}^{\nu-1} a_{i,J} w^i w_1^J$, where $a_{i,J}$ are rational integers.

LEMMA 4. *Let* K *be an integer. Consider* K *integers of* F,

$$A_k = \sum_{J=0}^{t-1} \sum_{\mu=0}^{\nu-1} b_{J,\mu,k} w^J w_1^\mu \quad (k = 1, \dots, K)$$

with $|b_{J,\mu,k}| \leqslant a$ *for all* J, μ, k. *Then there exist constants* λ *and* u *depending only on* w *and* w_1 *such that*

$$A = \prod_{k=1}^{K} A_k = \sum_{J=0}^{K(t+u)-1} \sum_{\mu=0}^{\nu-1} c_{J,\mu} w^J w_1^\mu$$

with $c_{J,\mu} \in \mathbf{Z}$, $|c_{J,\mu}| \leqslant (\lambda a t)^K$.

For the proof of the above lemma, see TIJDEMAN [7, p. 11].

§ 3. PROOF OF THEOREM 1. Denote by F the field obtained by adjoining to $\mathbf{Q}$ the numbers α_k, $\exp(\alpha_k \eta_J)$, $k = 0, 1, 2, 3$; $J = 0, 1, 2$. If $\alpha_0, \alpha_1, e^{\alpha_0 \eta_0}$ are algebraic, then $e^{\alpha_1 \eta_0} = (e^{\alpha_0 \eta_0})^{\alpha_1/\alpha_0}$, α_1/α_0 irrational, is transcendental by MAHLER's theorem [4]. (MAHLER [4] proved this theorem

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[1]) $H(F_N)$ denotes the maximum of the absolute values of the coefficients of F_N.

under a slightly stronger condition. This result can be found in ADAMS [1, p. 285]). So F can not be an algebraic extension of $\mathbf{Q}$. We assume that the transcendence degree of F over $\mathbf{Q}$ is one and we shall arrive at a contradiction.

There exists a transcendental number w such that F is a finite extension of $\mathbf{Q}(w)$. Further there exists $w_1 \in F$ which satisfies a monic irreducible polynomial of degree ν (say) over $\mathbf{Z}[w]$ and $F = \mathbf{Q}(w, w_1)$. Let T be an integer in F such that $T\alpha_k$, $T \exp(\alpha_k \eta_J)$, are all integers in F. Denote by $\lambda_1, \lambda_2, \lambda_3, \ldots$ positive constants >1 depending on $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \eta_0, \eta_1, \eta_2, P$ and the choice of w and w_1. Assume that N is a sufficiently large positive integer.

It is sufficient to prove the theorem when $|\alpha_i|_P < P^{-((1/P-1)+\varepsilon)}$, $0 \leqslant i < 4$, where $\varepsilon > 0$ is an arbitrary fixed constant. We set

$$K = [N^{4/3}], \quad p_1 = [N^{7/3}] + 1.$$

Consider the following auxiliary function

$$f(z) = \sum_{k_0=0}^{K-1} \sum_{k_1=0}^{K-1} \sum_{k_2=0}^{K-1} \sum_{k_3=0}^{K-1} \sum_{k=0}^{p_1-1} c_{k_0,k_1,k_2,k_3,k}\, w^k e^{(k_0\alpha_0 + k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3)z}, \quad |z|_P \leqslant P^{\varepsilon}$$

where $c_{k_0,k_1,k_2,k_3,k}$ are rational integers, not all zero, to be determined. Define

$$\lambda' = \frac{90}{\varepsilon \log P} + \frac{30}{\varepsilon}\left(\frac{1}{P-1} + \varepsilon\right), \quad S^* = [\lambda' N^{7/3}] + 1, \quad \mu = [3\lambda' N^{7/3}] > S^* + 12KN.$$

Denote

$$f_{s,n_0,n_1,n_2} = T^{\mu} f^{(s)}(n_0\eta_0 + n_1\eta_1 + n_2\eta_2).$$

For $0 \leqslant s < S^*$, $0 \leqslant n_i < N$, $i = 0, 1, 2$, apply lemma 4 to get

$$f_{s,n_0,n_1,n_2} = \sum_{q=0}^{\mu_1-1} \sum_{q_1=0}^{\nu-1} w^q w_1^{q_1} \sum_{k_0=0}^{K-1} \sum_{k_1=0}^{K-1} \sum_{k_2=0}^{K-1} \sum_{k_3=0}^{K-1} \sum_{k=0}^{p_1-1}$$
$$B_{s,n_0,n_1,n_2,k_0,k_1,k_2,k_3,k,q,q_1}\, c_{k_0,k_1,k_2,k_3,k}$$

where $\mu_1 \leqslant \lambda_1(\mu + p_1) \leqslant \lambda_3 N^{7/3}$ and B's are rational integers and they are in absolute value $\leqslant \exp(\lambda_2(\mu + p_1)\log N) \leqslant \exp(\lambda_4 N^{7/3} \log N)$. Define

$$S = [(32\lambda_3\nu)^{-1} N^{7/3}].$$

We want to choose $c_{k_0,k_1,k_2,k_3,k}$ in such a way that

$$f^{(s)}(n_0\eta_0 + n_1\eta_1 + n_2\eta_2) = 0, \quad 0 \leqslant s < S, \quad 0 \leqslant n_i < N; \quad i = 0, 1, 2.$$

For this consider the following equations

$$(1) \quad \sum_{k_0=0}^{K-1} \sum_{k_1=0}^{K-1} \sum_{k_2=0}^{K-1} \sum_{k_3=0}^{K-1} \sum_{k=0}^{p_1-1} c_{k_0,k_1,k_2,k_3,k} B_{s,n_0,n_1,n_2,k_0,k_1,k_2,k_3,k,q,q_1} = 0$$

for $0 \leqslant q < \mu_1$, $0 \leqslant q_1 < \nu$, $0 \leqslant s < S$, $0 \leqslant n_0, n_1, n_2 < N$.

This is a set of $\mu_1 \nu S N^3$ equations in $K^4 p_1$ variables $c_{k_0, k_1, k_2, k_3, k}$. Further

$$K^4 p_1 > {}^1/_{16} N^{23/3}$$

and

$$2\mu_1 \nu S N^3 \leqslant {}^1/_{16} N^{23/3}.$$

So

$$K^4 p_1 > 2\mu_1 \nu S N^3.$$

Hence by lemma 1, there exist rational integers $c_{k_0, k_1, k_2, k_3, k}$, not all zero, satisfying (1) and

$$|c_{k_0, k_1, k_2, k_3, k}| \leqslant 2K^4 p_1 \exp(\lambda_4 N^{7/3} \log N) \leqslant \exp(\lambda_5 N^{7/3} \log N).$$

(We shall make use of the Schnirelmann integral and we refer the readers to the paper of ADAMS [1, Appendix] for its definition and the properties).

Choose $R_1, R_2 \in T_P$ such that

$$1 < |R_1|_P < |R_2|_P < P^{\varepsilon}.$$

For every z such that $|z|_P = |R_1|_P$, we have

$$f(z) = \int_{0, R_2} \frac{f(\zeta)\zeta}{(\zeta - z)} \prod_{n_0=0}^{N-1} \prod_{n_1=0}^{N-1} \prod_{n_2=0}^{N-1} \left(\frac{z - n_0\eta_0 - n_1\eta_1 - n_2\eta_2}{\zeta - n_0\eta_0 - n_1\eta_1 - n_2\eta_2}\right)^S d\zeta.$$

We can assume that $|w|_P \leqslant 1$. Since w_1 is integral over $\mathbf{Z}[w]$, $|w_1|_P \leqslant 1$. Consequently $|T|_P \leqslant 1$. Also notice that for every z such that $|z|_P = |R_2|_P$, we have $|f(z)|_P \leqslant 1$. Hence for every z with $|z|_P = |R_1|_P$, we have

$$|f(z)|_P \leqslant |R_3|_P^{-N^3 S}, \quad |R_3|_P = |R_2 R_1^{-1}|_P > 1.$$

For $0 \leqslant s < S^*$ and $0 \leqslant n_0, n_1, n_2 < N$, we have

$$f^{(s)}(n_0\eta_0 + n_1\eta_1 + n_2\eta_2) = s! \int_{0, R_1} \frac{f(z) \cdot z}{(z - n_0\eta_0 - n_1\eta_1 - n_2\eta_2)^{s+1}} dz$$

and so for $s < S^*$, $n_i < N$, $i = 0, 1, 2$, we have

$$|f^{(s)}(n_0\eta_0 + n_1\eta_1 + n_2\eta_2)|_P \leqslant \max_{|z|_P = |R_1|_P} |f(z)|_P \leqslant |R_3|_P^{-N^3 S}.$$

And hence

$$|f_{s, n_0, n_1, n_2}|_P \leqslant |R_3|_P^{-N^3 S}.$$

Since w is transcendental and $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are linearly independent over $\mathbf{Q}$, $f(z)$ is not identically zero. Further

$$S^* N^3 > \lambda' N^{16/3} \geqslant \lambda' K^4.$$

So by lemma 2, there exists (s, n_0, n_1, n_2), $s < S^*$, $n_i < N$, $i = 0, 1, 2$ such that

$$f_{s, n_0, n_1, n_2} \neq 0.$$

Recall that

$$f_{s,n_0,n_1,n_2}=\sum_{q=0}^{\mu_1-1}\sum_{q_1=0}^{\nu-1} a_{q,q_1}w^q w_1^{q_1}$$

where a_{q,q_1} are rational integers and $|a_{q,q_1}|\leqslant \exp(\lambda_6 N^{7/3}\log N)$. Write

$$f_{s,n_0,n_1,n_2}=P_0(w, w_1) \text{ where } P_0(x, y)\in \mathbf{Z}[x, y].$$

Let $w_1, \ldots, w_\nu$ be the roots of the monic irreducible polynomial of w_1 over $\mathbf{Z}[w]$. Write

$$P(w)=\prod_{i=1}^{\nu} P_0(w, w_i).$$

Notice that $P(w)\neq 0$ and $P(x)\in \mathbf{Z}[x]$. Further notice that

$$\text{deg. } P\leqslant\lambda_7 N^{7/3}, \log H(P)\leqslant\lambda_8 N^{7/3}\log N$$

i.e.

$$\max(\text{deg. } P, \log H(P))\leqslant\lambda_9 N^{7/3}\log N.$$

Further

$$|P(w)|_P=|P_0(w,w_1)|_P\prod_{i=2}^{\nu}|P_0(w,w_i)|_P\leqslant|R_3|_P^{-N^3 S}\leqslant\exp(-\lambda_{10}^{-1}\log|R_3|_P N^{16/3}).$$

Set $f_N=\lambda_9 N^{7/3}\log N$ in lemma 3, $N\geqslant N_0$. We conclude that w is algebraic which is a contradiction. This completes the proof of theorem 1.

Proof of theorem 2. First observe that it is sufficient to prove the theorem when $|\alpha_i|_P<P^{-((1/P-1)+\varepsilon)}$ where $\varepsilon>0$ is an arbitrary fixed constant. Assume that the transcendence degree of F over $\mathbf{Q}$ is one. We set

$$K=[N^{2/3}],\ p_1=[N^{5/3}]+1,\ K_3=[N^{5/3}].$$

Define the following auxiliary function

$$f(z)=\sum_{k_0=0}^{K-1}\sum_{k_1=0}^{K-1}\sum_{k_2=0}^{K-1}\sum_{k_3=0}^{K_3-1}\sum_{k=0}^{p_1-1} c_{k_0,k_1,k_2,k_3,k}\, w^k z^{k_3} e^{(k_0\alpha_0+k_1\alpha_1+k_2\alpha_2)z}$$

where c's are rational integers, not all zero, to be determined. Define

$$\lambda'=\frac{90}{\varepsilon\log P}+\frac{30}{\varepsilon}\left(\frac{1}{P-1}+\varepsilon\right),$$

$$S^*=[\lambda' N^{5/3}]+1,\ \mu=[3\lambda' N^{5/3}],\ S=[(32\lambda_3\nu)^{-1}N^{5/3}]$$

where λ_3 is similar to the one which occurs in theorem 1. Denote $f_{s,n_0,n_1}=$ $=T^\mu f^{(s)}(n_0\eta_0+n_1\eta_1)$. ($T$ defined similarly as in theorem 1). And proceed as in theorem 1 and arrive at a contradiction.

Proof of theorem 3. Notice that F cannot be an algebraic extension of $\mathbf{Q}$. Assume that the transcendence degree of F over $\mathbf{Q}$ is one. It is sufficient to prove the theorem when $|\alpha_i|_P < P^{-((1/P-1)+\varepsilon)}$, $\varepsilon > 0$ is an arbitrary fixed constant. We set

$$K = [N^{5/2}], \ p_1 = [N^{7/2}] + 1.$$

Define the following auxiliary function

$$f(z) = \sum_{k_0=0}^{K-1} \sum_{k_1=0}^{K-1} \sum_{k_2=0}^{K-1} \sum_{k=0}^{p_1-1} c_{k_0, k_1, k_2, k}\, w^k e^{(k_0\alpha_0 + k_1\alpha_1 + k_2\alpha_2)z}$$

where c's are rational integers, not all zero, to be determined. Define

$$\lambda' = \frac{90}{\varepsilon \log P} + \frac{30}{\varepsilon}\left(\frac{1}{P-1} + \varepsilon\right),$$

$$S^* = [\lambda' N^{7/2}] + 1, \ \mu = [3\lambda' N^{7/2}], \ S = [(32\lambda_3 \nu)^{-1} N^{7/2}]$$

where λ_3 is similar to the one which occurs in theorem 1. Denote

$$f_{s, n_0, n_1, n_2, n_3} = T^\mu f^{(s)}(n_0\eta_0 + n_1\eta_1 + n_2\eta_2 + n_3\eta_3).$$

Proceed similarly as in theorem 1 and arrive at a contradiction.

Remark. We leave the following as an exercise to the readers: *Let $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in T_P$ with $|\alpha_i|_P < P^{-1/P-1}$ be linearly independent over* $\mathbf{Q}$. *Let $\eta_0, \eta_1, \eta_2, \eta_3 \in T_P$ with $|\eta_i|_P \leqslant 1$ be linearly independent over* $\mathbf{Q}$. *Then at least two of the numbers*

$$\exp(\alpha_k \eta_J), \ k = 0, 1, 2, 3, 4$$
$$J = 0, 1, 2, 3$$

are algebraically independent. As a consequence of this assertion, we mention: Let $\alpha \neq 1$, $|\alpha - 1|_P < P^{-1/P-1}$, be an algebraic number in T_P. Let $t \in T_P$, $|t|_P \leqslant 1$, be transcendental. Then at least two of the numbers

$$\alpha^t, \alpha^{t^2}, \alpha^{t^3}, \alpha^{t^4}, \alpha^{t^5}, \alpha^{t^6}, \alpha^{t^7}$$

are algebraically independent over $\mathbf{Q}$.

§ 4. *Appendix.* The purpose of this section is to prove lemma 2, which is a P-adic analogue of Tijdeman's result [8] on an upper bound of the number of zeros of a general exponential function in a disk.

Let $f(z)$ be a power series convergent for every z such that $|z|_P < R(>0)$. Choose $r \in T_P$ such that $|r|_P < R$. Let $\alpha_1, \ldots, \alpha_n \in T_P$, $|\alpha_i|_P < |r|_P$, $1 \leqslant i \leqslant n$, be a finite sequence of members of T_P. Then there exists a unique polynomial $P(z)$ of degree $< n$ such that $f(z) - P(z)$ is divisible by $(z-\alpha_1) \ldots \ldots (z-\alpha_n)$. We shall write

$$f(z) \equiv P(z) \ (\mathrm{mod.}\ \alpha_1, \ldots, \alpha_n).$$

LEMMA 5. *Let $f(z)$ be a power series convergent for every z such that $|z|_P<R$. Let $r\in T_P$ such that $|r|_P<R$. Let $\alpha_1, \ldots, \alpha_n$ be a finite sequence of distinct members of T_P such that $|\alpha_i|_P<|r|_P$, $1\leqslant i\leqslant n$. Let $P(z)$ be the polynomial such that*

$$f(z) \equiv P(z)(\text{mod.}\, \alpha_1, \ldots, \alpha_n).$$

Write

$$P(z)=\sum_{k=1}^{n} \pi_k(z-\alpha_1) \ldots (z-\alpha_{k-1}).$$

Then for $h=1, \ldots, n$, we have

$$\pi_h=\int_{O,r} \frac{f(z)\cdot z}{(z-\alpha_1) \ldots (z-\alpha_h)}\,dz. \tag{2}$$

PROOF. Consider

$$\int_{O,r} \frac{P(z)\cdot z}{(z-\alpha_1) \ldots (z-\alpha_h)}\,dz=\sum_{k=1}^{n} \pi_k \int_{O,r} \frac{(z-\alpha_1)\ldots(z-\alpha_{k-1})\cdot z}{(z-\alpha_1)\ldots(z-\alpha_h)}\,dz.$$

Now

$$\int_{O,r} \frac{(z-\alpha_1)\ldots(z-\alpha_{k-1})\cdot z}{(z-\alpha_1)\ldots(z-\alpha_h)}\,dz=\begin{cases}1 & \text{if } h=k\\ 0 & \text{otherwise}\end{cases}$$

So for $h=1, \ldots, n$, we have

$$\pi_h=\int_{O,r} \frac{P(z)\cdot z}{(z-\alpha_1) \ldots (z-\alpha_h)}\,dz. \tag{3}$$

Now note that, by definition,

$$\frac{f(z)-P(z)}{(z-\alpha_1) \ldots (z-\alpha_h)}$$

is a power series which is convergent for every z such that $|z|_P<R$. So

$$\int_{O,r} \frac{(f(z)-P(z))\cdot z}{(z-\alpha_1) \ldots (z-\alpha_h)}\,dz=0. \tag{4}$$

Combining (3) and (4), we get (2).

LEMMA 6. *Let $P(z)=\sum_{h=1}^{q} p_h z^{h-1}$ be a polynomial with coefficients in T_P. Then for any finite sequences $\alpha_1, \ldots, \alpha_n$ and $b_1, \ldots, b_n$ of T_P, we have*

$$|\sum_{k=1}^{n} b_k P(\alpha_k)|_P\leqslant \max_{h=1,\ldots,q} |p_h|_P\cdot \max_{J=1,\ldots,q} |\sum_{k=1}^{n} b_k \alpha_k^{J-1}|_P.$$

PROOF.

$$|\sum_{k=1}^{n} b_k P(\alpha_k)|_P=|\sum_{k=1}^{n} b_k \sum_{h=1}^{q} p_h \alpha_k^{h-1}|_P=|\sum_{h=1}^{q} p_h \sum_{k=1}^{n} b_k \alpha_k^{h-1}|_P\leqslant$$

$$\leqslant \max_{h=1,\ldots,q} |p_h|_P\cdot \max_{1\leqslant J\leqslant q} |\sum_{k=1}^{n} b_k \alpha_k^{J-1}|_P.$$

LEMMA 7. *Let $\alpha_1, \ldots, \alpha_n$ be a finite sequence of distinct numbers of T_P. Let $b_1, \ldots, b_n \in T_P$. Further assume that*

$$|\alpha_i|_P < P^{-((1/P-1)+4\varepsilon_1)}$$

where $\varepsilon_1 > 0$ is an arbitrary fixed constant. Define

$$G(z) = \sum_{k=1}^{n} b_k e^{\alpha_k z}, \; |z|_P \leqslant P^{4\varepsilon_1}.$$

Then for any $z_0 \in T_P$, $|z_0|_P \leqslant P^{\varepsilon_1}$, we have

$$|G(z_0)|_P \leqslant P^{((1/P-1)+4\varepsilon_1)(n-1)} \max_{J=1,\ldots,n} |G^{(J-1)}(0)|_P.$$

PROOF. The lemma is trivial if $z_0 = 0$. So assume that $z_0 \neq 0$. Define

$$f(z) = e^{zz_0}, \; |z|_P < P^{-((1/P-1)+\varepsilon_1)}.$$

Choose $r \in T_P$ such that

$$P^{-((1/P-1)+4\varepsilon_1)} < |r|_P < P^{-((1/P-1)+\varepsilon_1)}.$$

Let $P(z) = \sum_{k=1}^{n} \pi_k (z-\alpha_1) \ldots (z-\alpha_{k-1}) = \sum_{h=1}^{n} p_h z^{h-1}$ be the polynomial such that

$$f(z) \equiv P(z) (\mathrm{mod.}\, \alpha_1, \ldots, \alpha_n).$$

By lemma 5, we have

$$\pi_h = \int_{0,r} \frac{e^{zz_0} \cdot z}{(z-\alpha_1) \ldots (z-\alpha_h)} dz, \; h = 1, \ldots, n.$$

From here it follows that

$$|\pi_h|_P \leqslant |r|_P^{-(h-1)} \leqslant |r|_P^{-(n-1)}, \; h = 1, \ldots, n.$$

Clearly

$$\max_{h=1,\ldots,n} |p_h|_P \leqslant \max_{h=1,\ldots,n} |\pi_h|_P \leqslant |r|_P^{-(n-1)}.$$

Now

$$|G(z_0)|_P = |\sum_{k=1}^{n} b_k e^{\alpha_k z_0}|_P = |\sum_{k=1}^{n} b_k P(\alpha_k)|_P.$$

By lemma 6, we conclude that

$$\begin{aligned} |G(z_0)|_P &\leqslant \max_{h=1,\ldots,n} |p_h|_P \cdot \max_{J=1,\ldots,n} |\sum_{k=1}^{n} b_k \alpha_k^{J-1}|_P \\ &\leqslant |r|_P^{-(n-1)} \max_{J=1,\ldots,n} |G^{(J-1)}(0)|_P \\ &\leqslant P^{((1/P-1)+4\varepsilon_1)(n-1)} \max_{J=1,\ldots,n} |G^{(J-1)}(0)|_P. \end{aligned}$$

LEMMA 8. *Let $w_1, \ldots, w_n \in T_P$ and are distinct. Further assume that*

$$|w_i|_P < P^{-((1/P-1)+\varepsilon)}, \; i = 1, \ldots, n$$

where $\varepsilon > 0$ *is an arbitrary fixed constant. Assume that* $a_1, \ldots, a_n \in T_P$. *Define*

$$F(z) = \sum_{k=1}^{n} a_k e^{w_k z}, \quad |z|_P \leqslant P^{\varepsilon}.$$

Let $R_1, R_2 \in T_P$ *satisfying*

$$P^{(1/9)\varepsilon} < |R_1|_P < |R_2|_P < P^{(2/9)\varepsilon}.$$

Then

$$\max_{|z|_P \leqslant |R_2|_P} |F(z)|_P \leqslant \sqrt{2} P^{((1/P-1)+\varepsilon)(n-1)} \max_{|z|_P \leqslant |R_1|_P} |F(z)|_P. \tag{5}$$

PROOF. Choose $R_3 \in T_P$ such that $|R_2|_P < |R_3|_P < P^{(2/9)\varepsilon}$. Note that there exists $z_0 \in T_P$, $|z_0|_P = |R_1|_P$ such that

$$|F(z_0 R_3 R_1^{-1})|_P > (1/\sqrt{2}) \max_{|z|_P < |R_3|_P} |F(z)|_P. \tag{6}$$

Define for every z, $|z|_P \leqslant P^{(8/9)\varepsilon}$,

$$G(z) = F(zR_3 R_1^{-1}) = \sum_{k=1}^{n} a_k e^{\alpha_k z} \text{ where } \alpha_k = w_k R_3 R_1^{-1}.$$

Notice that α_k are distinct. Setting $\varepsilon_1 = {}^{(2/9)}\varepsilon$, we have for $|z|_P \leqslant P^{4\varepsilon_1}$

$$G(z) = \sum_{k=1}^{n} a_k e^{\alpha_k z}$$

with

$$|\alpha_k|_P = |w_k R_3 R_1^{-1}|_P \leqslant P^{-((1/P-1)+\varepsilon)+(1/9)\varepsilon} = P^{-((1/P-1)+4\varepsilon_1)}$$

and

$$|z_0|_P < |R_2|_P < P^{(2/9)\varepsilon} = P^{\varepsilon_1}.$$

All the assumptions of lemma 7 are satisfied and hence

$$|G(z_0)|_P \leqslant P^{((1/P-1)+(8/9)\varepsilon)(n-1)} \max_{1 \leqslant J \leqslant n} |G^{(J-1)}(0)|_P. \tag{7}$$

Now

$$G^{(J-1)}(0) = (R_3 R_1^{-1})^{J-1} F^{(J-1)}(0). \tag{8}$$

Further

$$\left\{ \begin{aligned} & F^{(J-1)}(0) = (J-1)! \int_{0,R_1} \frac{F(z)\cdot z}{z^J} dz \text{ and so (see [1, p. 298])} \\ & \qquad \max_{1 \leqslant J \leqslant n} |F^{(J-1)}(0)|_P \leqslant \max_{|z|_P \leqslant |R_1|_P} |F(z)|_P. \end{aligned} \right. \tag{9}$$

Combining (6), (7), (8) and (9), we get

$$\frac{1}{\sqrt{2}} \max_{|z|_P < |R_3|_P} |F(z)|_P \leqslant P^{((1/P-1)+(8/9)\varepsilon)(n-1)} |R_3 R_1^{-1}|_P^{n-1} \max_{|z|_P \leqslant |R_1|_P} |F(z)|_P.$$

Noting that $|R_2|_P < |R_3|_P$ and $|R_3 R_1^{-1}|_P < P^{(1/9)\varepsilon}$, we have

$$\max_{|z|_P \leqslant |R_2|_P} |F(z)|_P \leqslant \sqrt{2} P^{((1/P-1)+\varepsilon)(n-1)} \max_{|z|_P \leqslant |R_1|_P} |F(z)|_P.$$

REMARKS 1) The constant $\sqrt{2}$ on the right hand side of (5) can be replaced by any constant $u > 1$.

2) The assumption that α_i are distinct in lemma 5 and lemma 7 is not necessary. The same is true for w_i in lemma 8.

LEMMA 9 [1]. *Let $F(z)$ be defined as in lemma 4. Assume that $F(z)$ is not identically zero. Then the number of zeros (counted with their multiplicity) of $F(z)$ in $|z|_P \leqslant 1$ do not exceed*

$$\frac{90}{\varepsilon \log P} + \frac{30}{\varepsilon}\left(\frac{1}{P-1} + \varepsilon\right)(n-1).$$

PROOF. Let $F(z)$ vanish at $C_1, \ldots, C_N$, $|C_i|_P \leqslant 1$. (C_i need not be distinct, if C_i occurs i times, it means that $F(z)$ has an i-multiple zero at C_i). Define

$$g(z) = \frac{F(z)}{\prod_{n=1}^{N}(z - C_n)}.$$

Notice that $g(z)$ is a power series convergent for every z such that $|z|_P \leqslant P^\varepsilon$. Choose $R_1, R_2 \in T_P$ satisfying

$$P^{(1/9)\varepsilon} < |R_1|_P < P^{(1/6)\varepsilon}, \quad P^{(1/5)\varepsilon} < |R_2|_P < P^{(2/9)\varepsilon}. \tag{10}$$

Notice that

$$\max_{|z|_P = |R_1|_P} |g(z)|_P \leqslant |R_2|_P^{-N} \max_{|z|_P = |R_2|_P} |F(z)|_P. \tag{11}$$

Further notice that

$$\max_{|z|_P = |R_1|_P} |g(z)|_P \geqslant \tfrac{1}{2}|R_1|_P^{-N} \max_{|z|_P = |R_1|_P} |F(z)|_P. \tag{12}$$

Combining (10), (11) and (12), we get

$$P^{(1/30)\varepsilon N} \max_{|z|_P = |R_1|_P} |F(z)|_P \leqslant 2 \max_{|z|_P = |R_2|_P} |F(z)|_P.$$

By lemma 8, we get

$$P^{(1/30)\varepsilon N} \leqslant 4 P^{((1/P-1)+\varepsilon)(n-1)}.$$

Hence

$$N \leqslant \frac{90}{\varepsilon \log P} + \frac{30}{\varepsilon}\left(\frac{1}{P-1} + \varepsilon\right)(n-1)$$

and from here the lemma follows trivially.

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[1]) The method of deducing lemma 9 from lemma 8 was explained to me by Professor K. Ramachandra.

LEMMA 2. *Let $w_1, \ldots, w_l \in T_P$ be a finite sequence. Assume that $|w_i|_P < P^{-((1/P-1)+\varepsilon)}$, $\varepsilon > 0$ is an arbitrary fixed constant. Let $P_1(z), \ldots, P_l(z)$ be non-zero polynomials of degree $\sigma_1 - 1, \ldots, \sigma_l - 1$. Define*

$$F(z) = \sum_{k=1}^{l} P_k(z)\, e^{w_k z} \text{ for } |z|_P \leqslant P^{\varepsilon}$$

$$n = \sum_{k=1}^{l} \sigma_k.$$

Assume that $F(z)$ is not identically zero. Then the number of zeros of $F(z)$ in $|z|_P \leqslant 1$ (counted with their multiplicity) do not exceed

$$\frac{90}{\varepsilon \log P} + \frac{30}{\varepsilon}\left(\frac{1}{P-1} + \varepsilon\right)(n-1).$$

PROOF. The proof of lemma 2 is similar to that of lemma 9. One requires generalisations of lemma 6 and lemma 7 and the proof is omitted, since it needs no new ideas. (Use identities (6.3), (6.4) and (6.5) of [2, p. 8]).

Added in proof: For some developments in this connection, one can refer to the forthcoming paper 'Proprietes arithmetiques des valeurs de fonctions meromorphes algebriquement independantes' by WALDSCHMIDT, MICHEL in Vol. 23 of Acta Arithmetica.

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